EFFECT OF SURFACE TENSION ON THE ONSET OF CONVECTION IN A LAYER OF LIQUID WITH A FREE SURFACE

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The problem of stability of a plane horizontal layer of liquid heated from below is considered with surface tension at the upper surface taken into account. The problem is stated in section 1, proof of the existence of stability threshold is given in section 2, while section 3 concerns the construction of neutral curves by numerical methods and with the stabilizing effect of surface tension on the state of equilibrium.

The problem of convection onset in a liquid layer was considered in [1, 2] on the assumption that the free surface remains unperturbed. The critical Rayleigh number $R_* = 1100$ obtained there is valid for liquid layers of considerable thickness and strong gravitational fields, while the problem, as stated, excludes the effect of surface tension.

1. Equations and Boundary Conditions. A horizontal layer of liquid with free upper surface and bounded from below by a solid wall is placed in a gravitational field. Let us consider the problem of equilibrium stability when the temperature difference at the layer boundary surfaces is maintained constant. Equations of free convection are of the form [3]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\mathbf{v} = -\nabla p + \Delta \mathbf{v} - GT\gamma$$
$$\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = \frac{1}{P}\Delta T, \quad \operatorname{div}\mathbf{v} = 0 \qquad (1.1)$$
$$G = \frac{\beta g h^3 \delta}{\nu^2}, \quad P = \frac{\nu}{\varkappa}, \quad \gamma = \{0, 0, 1\}$$

Here $\mathbf{v}$ is the velocity vector, T is the temperature, p is the pressure, G is the Grashof number, P is the Prandtl number, β is the volumetric expansion coefficient of the liquid, g is the free-fall acceleration, δ is the difference of temperatures at the upper and lower surfaces, h is the mean thickness of the liquid layer assumed to be given and independent of time, ν is the coefficient of kinematic viscosity, χ is the coefficient of thermal diffusivity, and the x_3-axis is directed downward.

Surface tension at the free surface $x_3 = \varphi(x_1, x_2, t)$ is taken into consideration, and in accordance with [3] we assume fulfillment of the following conditions:

$$\tau_{ik}n_k - pn_i = \pi_0\delta_{3k} - S\left(\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2}\right)n_i + \frac{1}{F}\varphi n_i$$
$$\mathbf{v}\cdot\mathbf{n} = \left(1 + \frac{\partial\varphi^2}{\partial x_1} + \frac{\partial\varphi^2}{\partial x_2}\right)^{-1/2}\frac{\partial\varphi}{\partial t}, \quad T = \delta_2 = \text{const} \qquad (1.2)$$
$$F = \frac{\nu^2}{gh^3}, \quad S = \frac{a}{\rho g h^2}$$

Here π_0 is the external pressure, τ_{ik} is the stress tensor, n is a vector normal to the free surface, F and S are dimensionless parameters, a is the coefficient of surface tension, and ρ is density.

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We assume that at the wall $x_3 = 1$ the temperature and adhesion

$$T = \delta_1 = \text{const}, \qquad \mathbf{v} = 0 \tag{1.3}$$

are constant.

Let us assume that the functions $\mathbf{v}(\mathbf{x})$, $p(\mathbf{x})$, and $T(\mathbf{x})$, and $\varphi(x_1), x_2, t)$ are periodic in directions x_1 and x_2 with periods $L_1 = 2\pi/\alpha_1$ and $L_2 = 2\pi/\alpha_2$, respectively, and that the thickness of the liquid layer is constant, i.e.,

$$\int_0^{L_1}\int_0^{L_2} \varphi(x_1, x_2, t)\, dx_1\, dx_2 = 0 \tag{1.4}$$

and that there is no motion of the liquid in any horizontal direction

$$\int_0^{L_1}\int_0^{L_2}\int_{\varphi}^{1} v_i(x_1, x_2, x_3, t)\, dx_1\, dx_2\, dx_3 = 0 \quad (i = 1, 2) \tag{1.5}$$

2. The Linearized Problem. The following solution

$$\mathbf{v}_0 = 0, \quad T_0 = x_3, \quad p_0 = -\tfrac{1}{2}Gx_3^2, \quad \varphi_0 = 0 \tag{2.1}$$

of problem (1.1)-(1.3) corresponds to the equilibrium state of a liquid heated from below.

Using the small-oscillations method we linearize system (1.1)-(1.3) in the neighborhood of solution (2.1) and, assuming

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x})e^{\sigma t}, \quad T(\mathbf{x}, t) = T(\mathbf{x})e^{\sigma t}, \quad p(\mathbf{x}, t) = p(\mathbf{x})e^{\sigma t},$$
$$\varphi(x_1, x_2, t) = \varphi(x_1, x_2)e^{\sigma t}$$

eliminate time. As the result we obtain the spectral problem

$$\sigma\mathbf{v} = -\nabla p + \Delta\mathbf{v} - GT\gamma, \quad \sigma T + v_3 - \frac{1}{P}\Delta T = 0, \quad \operatorname{div}\mathbf{v} = 0 \tag{2.2}$$

$$x_3 = 0, \quad T = -\varphi, \quad v_3 = \sigma\varphi, \quad \tau_{13} = \tau_{23} = 0, \quad F\tau_{33} = \varphi - SF\left(\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2}\right)$$
$$x_3 = 1, \quad T = 0, \quad \mathbf{v} = 0 \tag{2.3}$$

As the stability threshold we consider those parameter values at which $\sigma = 0$ is a point of the spectrum of problem (2.2), (2.3). As to the oscillatory instability, it is only known that it is impossible for $F = 0$.

Assuming $\sigma = 0$ in (2.2) and (2.3), eliminating v_1, v_2, p, and φ, and separating variables

$$T(\mathbf{x}) = T(x_3)e^{i(\alpha_1 x_1 + \alpha_2 x_2)}, \quad v_3(\mathbf{x}) = v_3(x_3)e^{i(\alpha_1 x_1 + \alpha_2 x_2)}$$

we obtain the following eigenvalue problem:

$$L^2 v_3 = -\lambda\alpha^2 T, \qquad LT = v_3 \tag{2.4}$$

$$x_3 = 0, \quad v_3 = 0, \ Lv_3 = 0, \ \alpha^2(1 + \alpha^2\eta)\ T = \mu(DLv_3 - 2\alpha^2 Dv_3)$$
$$x_3 = 1, \ v_3 = 0, \ Dv_3 = 0, \ T = 0 \tag{2.5}$$
$$D = \frac{d}{dx_3}, \quad \alpha^2 = \alpha_1^2 + \alpha_2^2, \quad L = D^2 - \alpha^2, \quad \lambda = PG, \quad \mu = \frac{F}{P}, \quad \eta = SF$$

Problem (2.4), (2.5) differs from the similar problem in [4] by the positive factor

$$1 + \alpha^2\eta$$

The following theorem can be proved in the same manner as in [4].

Theorem 2.1. Problem (2.4), (2.5) has a simple eigenvalue λ_1 for any $\mu > 0$ and $\eta > 0$. A positive eigenfunction corresponds to that eigenvalue, and in a circle of radius λ_1 there are no other eigenvalues of problem (2.4), (2.5).

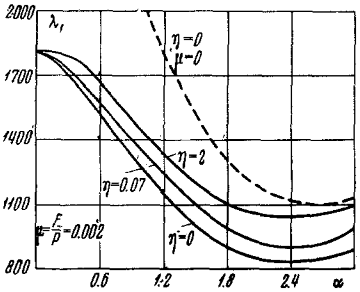

Fig. 1

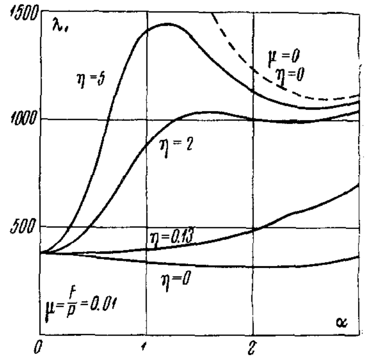

Fig. 2

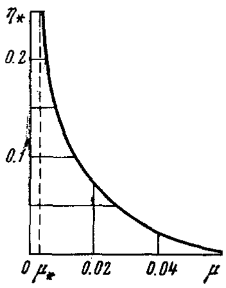

Fig. 3

For fixed α and μ, λ_1 is a monotonically increasing function of η and the estimates

$$\lambda_1(\alpha, \mu, \eta) \leqslant \frac{2\alpha \operatorname{sh}\alpha(\operatorname{sh}2\alpha - 2\alpha)(1+\alpha^2\eta)}{\mu[{}^3/_2 \operatorname{ch}\alpha(\operatorname{sh}2\alpha - 2\alpha) - \alpha^2(\alpha \operatorname{ch}\alpha + \operatorname{sh}\alpha)]}$$

$$\lambda_1(\alpha, \mu, \eta) \leqslant \lambda_1(\alpha, 0, 0)$$

are valid.

3. Numerical Calculation. The eigenvalues λ_1 of problem (2.4), (2.5) were numerically determined by the transcedental equation

$$\det A = 0$$

where $A = (a_{ik})$ is a fourth-order matrix whose elements are defined by the following expression:

$$a_{11} = -3(1+\alpha^2\eta) + a_{13}, \quad a_{12} = 6l(\lambda'+2)$$

$$a_{13} = l[a(\lambda'+2) + \sqrt{3}\, b(\lambda'-2)]$$

$$a_{14} = l\,[(\lambda'+2)b - \sqrt{3}a(\lambda'-2)], \quad a_{21} = \operatorname{Re} \operatorname{ch}\alpha c + 2e^a \cos\varphi_0$$

$$a_{22} = -\operatorname{Im} \operatorname{sh}\alpha c, \quad a_{23} = 2\operatorname{sh} a \cos\varphi_0, \quad a_{24} = 2\operatorname{ch} a \sin\varphi_0$$

$$a_{31} = -\operatorname{Re} \operatorname{ch}\alpha c, - 2e^{-a}\cos\varphi_1, \quad a_{32} = \operatorname{Im} \operatorname{sh}\alpha c$$

$$a_{33} = -e^a \cos\varphi_2 + e^{-a}\cos\varphi_1, \quad a_{34} = e^a \sin\varphi_2 + e^{-a}\sin\varphi_1$$

$$a_{41} = \operatorname{Re}(c \operatorname{ch}\alpha c) - 2e^a(a\cos\varphi_2 - b\sin\varphi_1), \quad a_{42} = \operatorname{Im}(c \operatorname{ch}\alpha c)$$

$$a_{43} = -e^a(a\cos\varphi_2 - b\sin\varphi_2) - e^{-a}(a\cos\varphi_1 + b\sin\varphi_1)$$

$$a_{44} = e^a(a\sin\varphi_2 + b\sin\varphi_1) - e^{-a}(a\sin\varphi_2 - b\cos\varphi_2)$$

$$\lambda' = (\lambda/\alpha^4)^{1/3}, \quad c = (1-\lambda)^{1/2}$$

$$a = [{}^1/_8(\sqrt{(\lambda'+2)^2 + 3\lambda'^2} + \lambda' + 2)]^{1/2}$$

$$b = [{}^1/_8(\sqrt{(\lambda'+2)^2 + 3\lambda'^2} - \lambda' - 2)]^{1/2}$$

$$\varphi_0 = \alpha b, \quad \varphi_1 = \alpha b + {}^2/_3\pi, \quad \varphi_2 = \alpha b - {}^2/_3\pi, \quad l = \lambda'\alpha^3\mu$$

Let us denote $\min_\alpha \lambda_1(\alpha, \mu, \eta)$ by $\lambda_*(\mu, \eta)$, and the particular value of α at which this minimum obtains by $\alpha_*(\mu, \eta)$. Curves of functions $\lambda_1(\alpha, \mu, \eta)$, calculated for fixed η and μ, are called neutral curves.

With the use of the perturbation theory it is possible to show that $\lambda_1(0, \mu, \eta)$ is independent of η, i.e., $\lambda_1(0, \mu, \eta) = \lambda_1(0, \mu, 0)$, and in accordance with [4]

$$\lambda_1(0, \mu, \eta) = {}^{40}/_{11}\, \mu^{-1}$$

of μ have a common point at $\alpha = 0$.

If this value is greater than $\lambda_*(0,0) = 1100.65$ [1], i.e., for $\mu > \mu_* = (40/11)\, \lambda^{*-1}(0,0)$, the stabilizing effect of surface tension consists in that $\lambda_*(\mu, \eta)$ and $\alpha_*(\mu, \eta)$ increase with increasing η and tend, respectively, to $\lambda_*(0,0)$ and $\alpha_*(0,0)$ as shown in Fig. 1.

For $\mu > \mu_*$ we have we have $\lambda_1 (0, \mu, \eta) < \lambda_* (0, 0)$ and there exists an $\eta = \eta_* (\mu)$ such that $\lambda_*(\mu, \eta) = 40/11\ \mu^{-1}$ and $\alpha_* (\mu, \eta_*) = 0$. Such neutral curves are shown in Fig. 2. The curve of function $\eta_*(\mu)$ is plotted in Fig. 3. Note that for sufficiently great η and $\mu > \mu_*$ the neutral curves have local minima close to the minimum of curve $\lambda_1 (\alpha, 0, 0)$.

The expression for parameter η contains the number F. Since for sufficiently thick liquid layers in strong gravitational fields this number is negligibly small, the effect of surface tension on the state of equilibrium of a liquid layer need not be taken into consideration. An example in which parameters μ and η must be taken into account is given below.

For a 1-mm-thick layer of glycerin $\mu = 0.007$ and $\eta = 30$, and convection must begin at Rayleigh number $R_* = 500$ with the wave number $\alpha = 0$, not at $R = 400$ and $\alpha = 2.2$, as stated in [4].

LITERATURE CITED

1. A. Pellew and R. V. Southwell, "On maintained convective motion in a fluid heated from below," Proc. Roy. Soc., Ser. A, vol. 176, pp. 312-342, 1940.
2. S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Clarendon Press, Oxford, 1961.
3. L. D. Landau and E. M. Lifshitz, Mechanics of Continuous Media [in Russian], 2-nd ed., Gostekhizdat, Moscow, 1954.
4. V. Kh. Izakson and V. I. Yudovich, "On the onset of convection in a liquid layer with a free surface," Izd. AN SSSR, MZhG [Fluid Dynamics], no. 4, 1968.